NUMERICAL ANALYSIS
TAKEHOME MIDTERM EXAM

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Abstract. Due by Friday, November 7, 2003.

1. Instructions

Browse to the website `www.saumag.edu/pbailey` and download the program `taylor.cpp`. Modify this program to obtain your results. Include printouts of the modified program(s) and their output when you hand in your exam.

The greater than sign redirects output in DOS and in UNIX. In order to coax a Visual C++ program to output to a file instead of to the screen, you can go go Project/Settings/Debug and enter "`> filename.txt`".

2. Discussion

Let $f : \mathbb{R} \to \mathbb{R}$. The n^{th} degree *Taylor approximation* of $f(x)$ is the polynomial function $T_n : \mathbb{R} \to \mathbb{R}$ defined by

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(x)}{i!} x^i.$$

The *sine over x integral* function, which is rumored to have important applications in electrical engineering, is defined by

$$\text{sox}(x) = \int_0^x \frac{\sin(t)}{t}\, dt.$$

Let $\text{sox}_n(x)$ denote the n^{th} Taylor approximation for $\text{sox}(x)$.

Since $\sin(x)$ and x are odd functions, $\frac{\sin(x)}{x}$ is an even function. The integral of an even function is odd, so sox is an odd function. We obtain the Taylor expansion of $\text{sox}(x)$ as follows: since

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots,$$

we have

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots.$$

Integrating term by term yields

$$\text{sox}(x) = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \dots.$$

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3. Problems

Problem 1. Create a function with the following prototype:
`double soxn(double x,int n);`
which returns $\text{sox}_n(x)$. Output a table of values for $\text{sox}_n(x)$ for
$n = 1, 3, 7, 15, 31, 63$ and for x is the range 0 to 4 at intervals of $h = 0.2$.

Problem 2. Create a function with the following prototype:
`double soxr(double x);`
which returns $\text{sox}_{63}(x) - 1$. Copying code from Program Set B Programs 1, 2, 3, find a solution to $\text{sox}_{63}(x) = 1$, accurate to 5 decimal places, in the interval $[0, 2]$. Use the midpoint as a seed point for the Newton method. In each case, how many iterations are required?

Problem 3. Create a function with the following prototype:
`double sox(double x);`
which returns $\text{sox}_{63}(x)$, which we view as an accurate approximation for $\text{sox}(x)$ near $x = 0$. Using a table constructed from $x \in \{0, 1, 2\}$, use Program Set C Program 3 to find the interpolation polynomial for $\text{sox}(x)$. That is, find its coefficients, accurate to 5 decimal places.

Problem 4. Apply Program Set C Program 4 to `sox` from Problem 3 to estimate the derivative at $x = 0$, using $n = 5$ and $h = 1$. Compare this to the derivative obtained from the interpolation polynomial, and to the derivative obtained by the well-known Calculus result.

Problem 5 (Optional)**.** Attempt to investigate the behavior of this function in the range $x \in [42, 45]$. You may have to increase n significantly. Are there more roots here? How many? Why?

4. Extra Credit

Problem 6. Let $f : \mathbb{Z}^+ \to \mathbb{Z}$ be an enumeration of the prime integers so that $f(1) = 2$, $f(2) = 3$, $f(3) = 5$, $f(4) = 7$, and so forth. Find $f(2^k)$ for $3 \leq k \leq 10$.

Problem 7. Let $a = 21945$ and $b = 34034$. Let $d = \gcd(a, b)$. Find d, and find $x, y \in \mathbb{Z}$ such that $ax + by = d$.

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